Calibration of forward LIBOR model

Math 622

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1 Introduction

From the lecture note on forward LIBOR, we've seen that the forward LIBOR rates for different maturity $T_j, 1 \leq j \leq n$ have the dynamics:

$$dL_\delta(t, T_j) = \gamma(t, T_j) L_\delta(t, T_j)\, d\widetilde{W}^{T_{j+1}},$$

where $\widetilde{W}^{T_{j+1}}$ is a Brownian motion under the T_{j+1} forward measure.

The financial products associated with these LIBOR rates are the caplets that pay $\delta(L_\delta(T_{j-1}, T_{j-1}) - K)^+$ at T_j. The market price of these caplets can be derived from the price of the caps:

$$\text{Caplet}_j(0) = \text{Cap}^m(0, j) - \text{Cap}^m(0, j-1).$$

On the other hand, from the model of the LIBOR rates, we can also derive, under the assumption that $\gamma(t, T_j)$ are determinstic, via Black-Scholes formula, the theoretical price of these caplets. We denote these prices by $\mathbf{Caplet}_j(0, \bar{\gamma}(T_{j-1}))$.

The obvious question is: can we build a model of these forward LIBOR rates so that

$$\text{Caplet}_j(0) = \mathbf{Caplet}_j(0, \bar{\gamma}(T_{j-1}))?$$

The answer is of course yes. Since $\mathbf{Caplet}_j(0, \bar{\gamma}(T_{j-1}))$ is a function of $\bar{\gamma}(T_{j-1}))$ we can choose a number γ_{j-1} so that the above equation holds:

$$\text{Caplet}_j(0) = \mathbf{Caplet}_j(0, \gamma_{j-1}).$$

γ_{j-1} is called the implied volatility of the LIBOR rate with maturity T_j. In general, we do not have an explicit formula for γ_{j-1}. The way to find γ_{j-1} is via numerical procedure, but it can be done.

Next we can construct a determinsitic function $\gamma(t, T_{j-1})$ so that

$$\int_0^{T_{j-1}} \gamma^2(t, T_{j-1}) = T_{j-1}\gamma_{j-1}.$$

There is much freedom in choosing $\gamma(t, T_{j-1})$ of course.

We may think the next step is just to construct n Brownian motions: $\widetilde{W}^{T_{j+1}}, 1 \leq j \leq n$ (question: how are they related?) and from which we can derive n LiBOR rates $L_\delta(t, T_j), 1 \leq j \leq n$ from which the Caplet price will match the market data. But this is missing some details.

First, we want to build a consistent model for $L_\delta(t, T_j), 1 \leq j \leq n$, beyond just matching the market data at time 0. Recall the definition of the LIBOR rates:

$$\begin{aligned} L_\delta(t, T_j) &= \frac{1}{\delta}\frac{B(t, T_j) - B(t, T_j + \delta)}{B(t, T_j + \delta)} \\ &= \frac{1}{\delta}\frac{B(t, T_j) - B(t, T_{j+1})}{B(t, T_{j+1})} \end{aligned}$$

So even without knowing the details, we should suspect that $L_\delta(t, T_j)$ and $L_\delta(t, T_{j+1})$ are related at some level. If we simulate $\widetilde{W}^{T_{j+1}}$ and $\widetilde{W}^{T_{j+2}}$ without regards to this relation, we're missing certain things.

Second, suppose starting out from the risk neutral measure $\widetilde{P}$, we have the dynamics of the bond $B(t, T_j)$ as

$$dB(t, T_j) = R(t)B(t, T_j)dt + \sigma^*(t, T_j)B(t, T_j)\widetilde{W}(t).$$

Note that there is only one Brownian motion $\widetilde{W}$ here, which is *independent* of T_j. (The choice of how many Brownian motions we put in is up to us, of course, but the point is that we use the same Brownian motions to model the dynamics of $B(t, T_j)$ for different T_j). So from what we learned from the change of numéraire section, the Brownian motion $\widetilde{W}^{T_{j+1}}$ are all related to $\widetilde{W}$ via the equation:

$$d\widetilde{W}^{T_j}(t) = d\widetilde{W}(t) + \sigma^*(t, T_j)\, dt.$$

Thus all Brownian motions $\widetilde{W}^{T_j}$ are related actually. So to model $L_\delta(t, T_j)$ properly, beyond determining the $\gamma(t, T_j)$ to match the market data, we also need to learn about the relations of $L_\delta(t, T_j)$. We will do so in the next section.

Finally, as the bond price $B(t, T_j)$ and LIBOR rates $L_\delta(t, T_j)$ are clearly related, we will see that by modeling the $L_\delta(t, T_j)$ properly, this will also give us a handle on how to model the volatility $\sigma^*(t, T_j)$ of the bonds and the (discounted) value of the bond $B(t, T_j)$ themselves. The details will be given in the third section.

2 Consistent forward LIBOR models - Relation among the $L(t, T_j)$

2.1 Relation among $\widetilde{W^{T_j}}$

Recall from section (1) that for every j, $d\widetilde{W}^{T_j}(t) = d\widetilde{W}(t) + \sigma^*(t, T_j)\, dt$. In particular, it follows from this that

$$d\widetilde{W}^{T_j}(t) = d\widetilde{W}^{T_{j+1}}(t) + [\sigma^*(t, T_j) - \sigma^*(t, T_{j+1})]\, dt \tag{1}$$

Next, from the dynamics of $L_\delta(t, T_j)$ that we derived before:

$$dL_\delta(t, T_j) = L_\delta(t, T_j) \left\{ \frac{1 + \delta L_\delta(t, T_j)}{\delta L_\delta(t, T_j)} [\sigma^*(t, T_{j+1}) - \sigma^*(t, T_j)] \right\} d\widetilde{W}^{T_{j+1}}(t).$$

This will be the same as the Black model $dL_\delta(t, T_j) = \gamma(t, T_j) L_\delta(t, T_j)\, d\widetilde{W}^{T_{j+1}}(t)$ only if

$$\gamma(t, T_j) = \frac{1 + \delta L_\delta(t, T_j)}{\delta L_\delta(t, T_j)} [\sigma^*(t, T_{j+1}) - \sigma^*(t, T_j)], \qquad t \leq T_j$$

or equivalently,

$$\sigma^*(t, T_{j+1}) - \sigma^*(t, T_j) = \frac{\delta L_\delta(t, T_j)}{1 + \delta L_\delta(t, T_j)} \gamma(t, T_j), \quad t \leq T_j. \tag{2}$$

Assume this is the case for all $j \leq n$. By combining this result with equation (1),

$$d\widetilde{W}^{T_j}(t) = d\widetilde{W}^{T_{j+1}}(t) - \frac{\delta L_\delta(t, T_j)}{1 + \delta L_\delta(t, T_j)} \gamma(t, T_j)\, dt \tag{3}$$

The significance of this equation is that *the processes $\sigma^*(t, T)$ no longer explicitly appear—everything is expressed in terms of the LIBOR rates themselves and their volatility functions $\gamma(t, T_j)$.*

By working backward with (3), $d\widetilde{W}^{T_j}(t)$ can be expressed in terms of $d\widetilde{W}^{T_{n+1}}(t)$ for all j. Indeed,

$$d\widetilde{W}^{T_n}(t) = d\widetilde{W}^{T_{n+1}}(t) - \frac{\delta L_\delta(t, T_n)}{1 + \delta L_\delta(t, T_n)} \gamma(t, T_n)\, dt.$$

But then

$$\begin{aligned} d\widetilde{W}^{T_{n-1}}(t) &= d\widetilde{W}^{T_n}(t) - \frac{\delta L_\delta(t, T_{n-1})}{1 + \delta L_\delta(t, T_{n-1})} \gamma(t, T_{n-1})\, dt \\ &= d\widetilde{W}^{T_{n+1}}(t) - \left[\frac{\delta L_\delta(t, T_n)}{1 + \delta L_\delta(t, T_n)} \gamma(t, T_n) + \frac{\delta L_\delta(t, T_{n-1})}{1 + \delta L_\delta(t, T_{n-1})} \gamma(t, T_{n-1}) \right] dt. \end{aligned}$$

Continuing further, and using what has just been derived,

$$\begin{aligned} d\widetilde{W}^{T_{n-2}}(t) &= d\widetilde{W}^{T_{n-1}}(t) - \frac{\delta L_\delta(t,T_{n-2})}{1+\delta L_\delta(t,T_{n-2})}\gamma(t,T_{n-2})\,dt \\ &= d\widetilde{W}^{T_{n+1}}(t) - \left[\sum_{i=n-2}^{n} \frac{\delta L_\delta(t,T_i)}{1+\delta L_\delta(t,T_i)}\gamma(t,T_i)\right] dt. \end{aligned}$$

Clearly, this will yield for general $j \leq n$ that

$$d\widetilde{W}^{T_j}(t) = d\widetilde{W}^{T_{n+1}}(t) - \left[\sum_{i=j}^{n} \frac{\delta L_\delta(t,T_i)}{1+\delta L_\delta(t,T_i)}\gamma(t,T_i)\right] dt. \tag{4}$$

The significance of this equation, compare with (3) is that *now all $\widetilde{W^{T_j}}$ is written in terms of $\widetilde{W^{T_{n+1}}}$. Thus, instead of generating n Brownian motions, we only need to generate one Brownian motion $\widetilde{W^{T_{n+1}}}$. This is consistent with what we mentioned before that we started out with only one Brownian Motion under risk neutral measure $\widetilde{W}$.*

2.2 The relation among the $L(t, T_j)$ - Their construction

Now we can write down a coherent system of equations for the LIBOR forward rates . First of all, Black's model for $j = n$ gives

$$dL_\delta(t,T_n) = L_\delta(t,T_n)\gamma(t,T_n)\,d\widetilde{W}^{T_{n+1}}(t), \qquad t \leq T_n. \tag{5}$$

Next, for arbitrary $j < n$, $dL_\delta(t,T_j) = L_\delta(t,T_j)\gamma(t,T_j)\,d\widetilde{W}^{T_{j+1}}(t)$, and so

$$dL_\delta(t,T_j) = L_\delta(t,T_j)\gamma(t,T_j)\left[-\sum_{i=j+1}^{n} \frac{\delta L_\delta(t,T_i)}{1+\delta L_\delta(t,T_i)}\gamma(t,T_i) + d\widetilde{W}^{T_{n+1}}(t)\right], \qquad t \leq T_j \tag{6}$$

This system of equations makes no reference to the original risk-neutral HJM model. In fact, it can stand alone as its own model. By working backwards on this set of equations using standard theorems, one can prove that it generates a consistent model for caplets of all maturities up to T_{n+1}, without assuming the prior existence of an HJM model for $B(t,T)$. We state this result and summarize the forward LIBOR model in the following theorem. We will not give the proof, but it involves only techniques we know already from Chapter 9.

Theorem 1. *Let there be given a probability space with measure $\widetilde{\mathbf{P}}^{T_{n+1}}$ supporting a Brownian motion $\widetilde{W}^{T_{n+1}}$. Then there exists a unique solution $L_\delta(t,T_1),\ldots,L_\delta(t,T_n)$*

to the system of equations (5)–(6). If the measures $\widetilde{\mathbf{P}}^{T_j}$, $j = n, n-1, \ldots, 1$ *are defined recursively by*

$$\widetilde{\mathbf{P}}^{T_j}(A) = \tilde{E}^{T_{j+1}}\left[\mathbf{1}_A \frac{1+\delta L_\delta(T_j,T_j)}{1+\delta L_\delta(0,T_j)}\right],$$

and the processes $\widetilde{W}^{T_j}(t)$, $1 \le j \le n$, *are defined recursively by*

$$d\widetilde{W}^{T_j}(t) = d\widetilde{W}^{T_{j+1}}(t) - \frac{\delta L_\delta(t,T_j)}{1+\delta L_\delta(t,T_j)}\gamma(t,T_j)\,dt,$$

then $\widetilde{W}^{T_j}$ *is a Brownian motion under* $\widetilde{\mathbf{P}}^{T_j}$ *for each* $j \le n$ *and*

$$dL_\delta(t,T_j) = L_\delta(t,T_j)\gamma(t,T_j)\,d\widetilde{W}^{T_{j+1}}(t), \quad \textit{for each } j \le n.$$

3 Construction the T_j-Maturity Discounted Bonds

3.1 Construction of $\sigma^*(t,T_j)$

The above theorem does not give us a HJM model, which is defined in terms of functions $\sigma^*(t,T_j)$ on the risk-neutral probability for prices denominated in the domestic currency. This is done in Shreve on pages 444-447. We will only outline the main idea here.

With the deterministic functions $\gamma(t,T_j)$ in hand, we can construct the functions $\sigma^*(t,T_j)$ that are consistent with $\gamma(t,T_j)$

$$\sigma^*(t,T_{j+1}) - \sigma^*(t,T_j) = \frac{\delta L_\delta(t,T_j)}{1+\delta L_\delta(t,T_j)}\gamma(t,T_j) \quad t \le T_j.$$

By writing this as

$$\sigma^*(t,T_{j+1}) = \sigma^*(t,T_j) + \frac{\delta L_\delta(t,T_j)}{1+\delta L_\delta(t,T_j)}\gamma(t,T_j), \quad t \le T_j. \tag{7}$$

we see that $L_\delta(t,T_j)$, $\gamma(t,T_j)$, and $\gamma^*(t,T_j)$ *determine* $\sigma^*(t,T_{j+1})$ at least for $t \le T_j$. This leads to a recursive procedure for defining $\sigma^*(t,T_j)$. We outline the procedure of construction here:

1. Choose $\sigma^*(t,T_1)$ for $0 \le t \le T_1$. The only constraint is

$$\lim_{t\to T_1} \sigma^*(t,T_1) = 0.$$

2. Construct $\sigma^*(t, T_2)$ for $0 \le t \le T_1$ (note the time interval) using the relation

$$\sigma^*(t, T_{j+1}) = \sigma^*(t, T_j) + \frac{\delta L_\delta(t, T_j)}{1 + \delta L_\delta(t, T_j)} \gamma(t, T_j), \quad t \le T_j.$$

3. Choose $\sigma^*(t, T_2)$ for $T_1 \le t \le T_2$ (again note the time interval). The only constraint is

$$\lim_{t \to T_2} \sigma^*(t, T_2) = 0.$$

4. Repeat this procedure to construct $\sigma^*(t, T_j)$ for $j \ge 3$.

Observe that in the above procedure, we had freedom to construct $\sigma^*(t, T_j)$ on the interval $T_{j-1} \le t \le T_j$ subject to the only constraint

$$\lim_{t \to T_j} \sigma^*(t, T_j) = 0.$$

Thus there is also much freedom in constructing $\sigma^*(t, T_j)$.

3.2 Construction the T_j-Maturity Discounted Bonds

Now that we have constructed $\sigma^*(t, T_j)$, the dynamics of the bond $B(t, T_j)$ under the risk neutral measure $\widetilde{P}$ is straightforward:

$$dB(t, T_j) = R(t)B(t, T_j)dt - \sigma^*(t, T_j)B(t, T_j)d\widetilde{W}(t).$$

Since we constructed the LIBOR rate under the forward measure $\widetilde{P}^{T_{n+1}}$ and the Brownian motion $\widetilde{W}^{T_{n+1}}$, it's also convenient to write the dynamics of $B(t, T_j)$ using these as well:

$$dB(t, T_j) = R(t)B(t, T_j)dt + \sigma^*(t, T_j)\sigma^*(t, T_{n+1})B(t, T_j)dt - \sigma^*(t, T_j)B(t, T_j)d\widetilde{W}^{T_{n+1}}(t).$$

Lastly, since we haven't constructed $R(t)$, it is better to write the dynamics of the discounted bond price instead:

$$d\Big(D(t)B(t, T_j)\Big) = \sigma^*(t, T_j)\sigma^*(t, T_{n+1})D(t)B(t, T_j)dt - \sigma^*(t, T_j)D(t)B(t, T_j)d\widetilde{W}^{T_{n+1}}(t).$$

We need the initial conditions to generate the bonds. They can be obtained from the LIBOR rates we have constructed as well:

$$D(0)B(0, T_j) = B(0, T_j) = \prod_{i=0}^{j-1} \frac{B(0, T_{i+1})}{B(0, T_i)} = \prod_{i=0}^{j-1} \Big(1 + \delta L_\delta(0, T_i)\Big)^{-1}.$$

The verification that our construction is consistent: $D(t)B(t,T_j)$ is a martingale under $\widetilde{P}$ is stated in Shreve's Theorem 10.4.4. (The only subtle point is we start out modeling under the forward measure $\widetilde{P}^{T_{n+1}}$. So we need to *define* the risk neutral measure $\widetilde{P}$ from $\widetilde{P}^{T_{n+1}}$. After that the verification is straightforward.)